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FINITE GROUPS WITH *s-c*-PERMUTABLY EMBEDDED SUBGROUPS

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Abstract

A subgroup H of a group G is said to be s-c-permutably embedded in G if for each prime $p \in \pi(H)$, every Sylow p-subgroup of H is a Sylow p-subgroup of some s-conditionally permutable subgroups of G. In this paper, we obtain some results about the s-c-permutably embedded subgroups and use them to determine the structure of some groups.

1. Introduction

In [6, 7] Guo et al. and in [8] Huang and Guo defined c-permutability and s-c-permutability of a subgroup of a finite group. They say that a

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subgroup H of a group G is conditionally permutable (or in brevity, c-permutable) in G if for any subgroup T of G, there exists some $x \in G$ such that $HT^x = T^x H$. A subgroup H of a group G is said to be s-conditionally permutable (or in breivity, s-c-permutable) in G if for every Sylow subgroup P of G, there exists $x \in G$ such that $PH^x = H^x P$. Using the new idea, people have obtained a series of elegant results on the structure of groups (see [4-8]). As a development, in [2] we introduced the following concept of s-c-permutably embedded subgroups:

Definition [2]**.** *A* subgroup H of a group G is said to be s-c-permutably embedded in G if every Sylow subgroup of H is a Sylow subgroup of some s-conditionally permutable subgroups of G.

In [2], one has seen that every s-c-permutable subgroup of G is an s-c-permutably embedded subgroup of G, but the converse is not true. By using the s-c-permutably embedded subgroups, we have obtained some conditions of supersolvability of finite groups. The purpose of this paper is to go further into the influence of s-c-permutably embedded subgroups on the structure of finite groups. Some new results are obtained.

Throughout, all groups are assumed to be finite groups. The unexplained notations and terminology are standard. The reader is referred to Huppert [9] or Guo [3].

2. Preliminaries

We denote $M < \cdot G$ to indicate that M is a maximal subgroup of G. Note that an automorphism of a group G that leaves every subgroup invariant is called a *power automorphism* (see [12, p. 389]). A class $\mathfrak{F}$ of groups is called a *formation* if it is closed under homomorphic image and every group G has a smallest normal subgroup (called $\mathfrak{F}$-*residual* and denoted by $G^{\mathfrak{F}}$) with quotient is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. We use $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{S}$ to denote the formation of all nilpotent, supersoluble and soluble groups, respectively.

For the sake of convenience, we list here some known results as lemmas, which will be used in this paper.

Lemma 2.1 [2, Lemma 2.2]. *Suppose that G is a group,* $K \lhd G$ *and* $H \leq G$. *Then:*

(1) *If H is s-c-permutably embedded in G, then* HK/K *is s-c-permutably embedded in* G/K.

(2) *If* $K \leq H$ *and* H/K *is s-c-permutably embedded in* G/K, *then H is s-c-permutably embedded in G.*

(3) *If* HK/K *is s-c-permutably embedded in* G/K *and* $(|H|, |K|) = 1$, *then H is s-c-permutably embedded in G.*

(4) *If H is s-c-permutably embedded in G, then* $H \cap K$ *is s-c-permutably embedded in K. In particular, if* $H \leq K \leq G$ *and H is s-c-permutably embedded in G, then H is s-c-permutably embedded in K.*

Lemma 2.2 [2, Corollary 3.2.1]. *Let G be a soluble group. If every maximal subgroup of each Sylow subgroup of G is s-c-permutably embedded in G. Then G is supersoluble.*

Lemma 2.3. *Let G be a group and P be a subgroup of G and P be a subgroup contained in* $O_p(G)$. *If P is s-c-permutably embedded in G, then P is s-c-permutable in G.*

Proof. Obviously, P is a subnormal subgroup of G. Since P is *s*-*c*-permutably embedded in G, there exists an *s*-conditionally permutable subgroup A of G such that P is a Sylow p-subgroup of A. Suppose q is an arbitrary prime divisor of $|G|$. Then there exists a Sylow q-subgroup G_q of G such that $AG_q = G_qA$. If $p = q$, then $P \leq G_p$ and so $PG_p = G_pP$. If $p \neq q$, then P is a subnormal Hall subgroup of $AG_q = G_qA$. It follows that P is normal in AG_q and consequently $PG_q = G_qP$. Thus P is *s*-conditionally permutable in G.

Lemma 2.4 [11, Lemma 2.5]. *Let G be a finite group and p be a prime such that* $(|G|, p^2 - 1) = 1$. *If* G/L *is p-nilpotent and* $p^3 \nmid |L|$, *then G is p-nilpotent.*

3. Main Results

Theorem 3.1. *Let G be a p-soluble subgroup. Then G is p-supersoluble if and only if for any non- Frattini p-chief factor H/K, there exists a maximal subgroup P_1 of a Sylow p-subgroup of G such that P_1 is s-c-permutably embedded in G and $H/K \not\subseteq P_1K/K$.*

Proof. We first prove the necessity part.

Let G be a p-supersoluble group and H/K be a non-Frattini p-chief factor of G. Then $|H/K| = p$ and there exists a maximal subgroup M of G such that $H \subset M$ and $K \leq M$. Clearly, $|G : M| = p$. Let P_1 be a Sylow p-subgroup of M. Then P_1 is a maximal subgroup of a Sylow p-subgroup of G and $H/K \not\subseteq P_1K/K$. Since $|G : M| = p$, M is clearly s-c-permutable in G. Consequently P_1 is s-c-permutably embedded in G.

Now we prove the sufficiency part. Assume that the result is false and let G be a counterexample of minimal order.

Let N be a minimal normal subgroup of G and $(H/N)/(K/N)$ be an arbitrary non-Frattini p-chief factor of G/N. Then H/K is a p-chief factor of G/K. If $H/K \subseteq \Phi(G/K) = \bigcap_{K \subseteq M <\cdot G} M/K$, then $H \subseteq \bigcap_{K \leq M <\cdot G} M$ and hence $(H/N)/(K/N) \subseteq \Phi((G/N)/(K/N))$, a contradiction. This shows that H/K is a non-Frattini p-chief factor of G. By the hypothesis, there exists a maximal subgroup P_1 of a Sylow p-subgroup of G such that P_1 is s-c-permutably embedded in G and $H/K \not\subseteq P_1K/K$. Then by Lemma 2.1, P_1N/N is s-c-permutably embedded in G/N and clearly $(H/K)/(K/N) \not\subseteq (P_1K/N)(K/N)$. Hence by the choice of $G, G/N$ is p-supersoluble. If $N \subseteq \Phi(G)$ or $N \subseteq O_{p'}(G)$, then clearly G is p-supersoluble. Hence we can assume, without loss of generality, that $N \not\subseteq \Phi(G)$ and N is an elementary abelian p-group. By the hypothesis, there exists a maximal subgroup P_1 of some Sylow p-

subgroup of G such that $N \not\subseteq P_1$ and P_1 is s-c-permutably embedded in G. Hence there exists an s-conditionally permutable subgroup A of G such that P_1 is a Sylow p-subgroup of A. This means that for any prime divisor q of $|G|$, there exists a Sylow q-subgroup Q of G such that $AQ = QA$. Let $N_1 = N \cap P_1$. Since $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = p$, N_1 is a maximal subgroup of N. Suppose that $q \neq p$. Since $N_1 = N \cap P_1 \leq N \cap A \leq N \cap AQ \leq N$ and N_1 is the maximal subgroup of N, $N_1 = N \cap AQ$ or $N = N \cap AQ$. If $N = N \cap AQ$, then $N \subseteq P_1$, a contradiction. Thus $N_1 = N \cap AQ$ and hence $Q \subseteq N_G(N_1)$. On the other hand, since $N_1 = N \cap P_1 \trianglelefteq P_1$, $N_1 \trianglelefteq P_1 N$. This implies that $N_1 \trianglelefteq G$. Thus $N_1 = 1$ by the minimal choice of N. It follows that $|N| = p$ and consequently G is p-supersoluble. The final contradiction completes the proof.

Corollary 3.1.1 [13, Theorem 3.1]**.** Let G be *a* p-soluble group. Then G is p-supersoluble if and only if for each prime divisor p of $|G|$ and each non-Frattini p-chief factor H/K, there exists a maximal subgroup P_1 of *a* Sylow p-subgroup P such that P_1 is s-conditionally permutable in G and $H/K \not\subseteq P_1K/K$.

Theorem 3.2. *Let G be a p-soluble group, p be the prime such that $(|G|, p^2 - 1) = 1$. If there exists a normal subgroup H of G such that G/H is p-nilpotent and every 2-maximal subgroup of any Sylow p-subgroup of H is s-c-permutably embedded in G, then G is p-nilpotent.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. We proceed the proof via the following steps.

(1) $O_{p'}(G) = 1$

Suppose that $O_{p'}(G) \neq 1$. Then by Lemma 2.1, we can see that $G/O_{p'}(G)$ (with respect to $HO_{p'}(G)/O_{p'}(G)$) satisfies the hypothesis.

Hence, $G/O_{p'}(G)$ is p-nilpotent by the choice of G. This implies that G is p-nilpotent, a contradiction.

(2) G has a unique minimal normal subgroup L such that G/L is p-nilpotent and $\Phi(G) = 1$.

Let L be a minimal normal subgroup of G. Then $L \leq H$ or $L \cap H = 1$. Hence, by Lemma 2.1, we can see that G/L (with respect to HL/L) satisfies the hypothesis. The minimal choice of G implies that G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, G has a unique normal subgroup, N say and $\Phi(G) = 1$.

(3) $G = [L]M$, where $p^3 \mid |L|$, M is nilpotent and $L = O_p(G) = F(G) = C_G(L)$.

Since G is p-soluble, $L \subseteq O_p(G)$ by (1). Hence, $L \subseteq O_p(G) \subseteq F(G) \subseteq C_G(L)$. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = LM$. Obviously, L is abelian. Hence, $L \cap M \trianglelefteq LM = G$ and so $L \cap M = 1$. Consequently, $G = [L]M$. Since $C_G(L) \cap M \trianglelefteq LM = G, C_G(L) \cap M = 1$ by (2). Hence $C_G(L) = C_G(L) \cap LM = L$. By (2), $M \cong G/L$ is p-nilpotent. By Lemma 2.4, $p^3 \mid |L|$.

(4) Final contradiction.

Let M_p be a Sylow p-subgroup of M and G_p be a Sylow p-subgroup of G containing M_p. Then $G_p = LM_p$ and $|G_p : M_p| = |L| \geq p^3$. Hence there exists a 2-maximal subgroup P_1 of G_p such that $M_p \leq P_1$. Let $P_2 = P_1 \cap H$. Obviously, $H_p = G_p \cap H$ is a Sylow p-subgroup of H and $P_2 = P_1 \cap H = P_1 \cap H_p$. Since $G_p = LM_p = LP_1 = H_pP_1, |H_p : P_2| = |H_pP_1 : P_1| = p^2$. This implies that $P_2 = P_1 \cap H$ is a 2-maximal subgroup of H_p. By the hypothesis, P_2 is s-c-permutably embedded in G. Hence there exists an s-conditionally permutable subgroup A of G such that P_2 is a

Sylow p-subgroup of A. Hence for arbitrary $q \in \pi(G)$ and $q \neq p$, there exists a Sylow q-subgroup G_q of G such that $AG_q = G_q A$. Let $L_1 = L \cap P_2$. Then $|L : L_1| = |L : L \cap P_2| = |LP_2 : P_2| = |L(P_1 \cap H) : P_2| = |LP_1 \cap H : P_2| = |H_p : P_2| = p^2$. This implies that L_1 is a 2-maximal subgroup of L. Since $L_1 = L \cap P_2 = L \cap A = L \cap AQ$, $L_1 \trianglelefteq AQ$ and so $Q \subseteq N_G(L_1)$. On the other hand, since $L \cap P_2 = L \cap H \cap P_1 \trianglelefteq P_1$ and $L \cap P_2 \trianglelefteq L$ (since L is abelian), $L_1 \trianglelefteq G_p$. This shows that $L_1 \trianglelefteq G$. But since L is a minimal normal subgroup of G, $L_1 = 1$, which contradicts the fact that $p^3 \mid |L|$. This contradiction completes the proof.

Theorem 3.3. *Let G be a soluble group. Then every maximal subgroup of every Sylow subgroup of G is s-c-permutably embedded in G if and only if*

(1) $G = [H]K$, *where H is a nontrivial nilpotent normal Hall subgroup of G.*

(2) *For every Sylow subgroup S of H, every element of K induces a power automorphism on* $S / \Phi(S)$ *by a conjugate actions.*

(3) *For every* $p \in \pi(K)$, *every maximal subgroup of a Sylow p-subgroup P of G is s-c-permutably embedded in G.*

Proof. The necessity part: By Lemma 2.2, G is supersoluble. Assume that p is the largest prime divisor of $|G|$ and P is a Sylow p-subgroup of G. Then $P \trianglelefteq G$. Let H be the products of all normal Sylow subgroups of G. Then H is a normal nilpotent Hall subgroup of G. By Schur-Zassenhaus Theorem, H has a complement K in G. Hence (1) holds.

Assume that S_1 is a maximal subgroup of some Sylow subgroup S of H. Then by the hypothesis, S_1 is s-c-permutably embedded in G. By Lemma 2.3, S_1 is s-c-permutable in G. Let q be an arbitrary prime divisor of $|G|$ with $p \neq q$. Then there exists a Sylow q-subgroup Q such that

$S_1Q = QS_1$. Since $S \trianglelefteq G$, S_1 is a subnormal Hall subgroup of S_1Q and so $Q \subseteq N_G(S_1)$. It follows that $S_1 \trianglelefteq G$ since $S_1 \trianglelefteq S$. Hence every maximal subgroup of $S / \Phi(S)$ is normal in $G / \Phi(S)$. Assume that $1 \neq \overline{x} = x\Phi(S) \in S / \Phi(S) = \overline{S}$. By [3, Theorem 1.8.17], we know that $\overline{S} = \langle \overline{a}_1 \rangle \times \langle \overline{a}_2 \rangle \times \cdots \times \langle \overline{a}_n \rangle$, where $\langle \overline{a}_i \rangle$ is a minimal normal subgroup of $\overline{G}$. We may assume that $\overline{a}_1 = \overline{x}$ and $\overline{a}_i = a_i\Phi(S)$. Let $P_i = \langle \overline{a}_1 \rangle \times \cdots \langle \overline{a}_{i-1} \rangle \times \langle \overline{a}_{i+1} \rangle \times \cdots \times \langle \overline{a}_n \rangle$. Then $\overline{P_i}$ is a maximal subgroup of $\overline{S}$ and $\overline{a}_1 = \bigcap_{i=2}^n \overline{P_i}$. Since $\overline{P_i} \lhd G / \Phi(S)$, $\overline{P_i}$ is K-invariant. It follows that $\bigcap_{i=2}^n \overline{P_i} = \langle \overline{a}_1 \rangle = \langle \overline{x} \rangle$ is K-invariant. Similarly, $\langle \overline{a}_i \rangle$ is K-invariant for all $i \in \{2, \cdots, n\}$. Since $S / \Phi(S)$ is an elementary abelian group, every subgroup of $S / \Phi(S)$ is the product of some $\langle \overline{a}_i \rangle, i = 1, 2 \cdots, n$. This shows that every element of K induces a power automorphism on $S / \Phi(S)$ by a conjugate actions. Thus (2) holds.

(3) is obvious. The necessary part is proved.

The sufficiency part: Assume that P_1 is a maximal subgroup of some Sylow p-subgroup P of G. If $p | |H|$, then $P \leq H$ by (1). It follows that $P_1 \trianglelefteq G$ by (2) and so P_1 is s-c-permutably embedded in G. Now assume that $p| \, |K|$. Then by (3), P_1 is s-c-permutably embedded in G. Thus the sufficiency holds.

It is well known that the product of two supersoluble subgroups is not necessarily supersoluble. Bear [1] proved that the product $G = AB$ of two normal supersoluble groups A and B is supersoluble if G' is nilpotent. Guo et al. [7] proved that the product $G = AB$ of two supersoluble subgroups is supersoluble if G' is nilpotent, A is completely c-permutable with every subgroup of B and B is completely c-permutable with every subgroup of A. Liu et al. [10] proved that A group G is supersoluble if and only if $G = AB$ is the product of two supersoluble subgroups A and B such that every normal subgroup of A is permutable with every Sylow subgroup of B and every normal subgroup of B is permutable with every Sylow subgroup of A. Now we give a new criterion of supersolubility of a product of two supersoluble subgroups by using the s-c-permutably embedded subgroups.

Theorem 3.4. *Let $G = AB$ be a product of two supersoluble subgroups and G' is nilpotent. If every primary cyclic subgroup of A and of B are s-c-permutably embedded in G, then G is supersoluble.*

Proof. Assume that the assertion is false and let G be a counterexample with minimal order. We proceed with our proof as follows:

(1) G is soluble (This is clear since G' is nilpotent).

(2) If N is a minimal normal subgroup of G, then G/N is supersoluble.

Indeed, $G/N = (AN/N)(BN/N)$ and $(G/N)' = G'N/N$ is nilpotent. Let T/N be a primary cyclic subgroup of AN/N. Obviously, $TN = \langle x \rangle N$, where $\langle x \rangle$ is a primary cyclic group of T. Since $T \leq AN$, $x = an$, where $a \in A, n \in N$ and a is a primary element. Hence $\langle x \rangle N = \langle a \rangle N$. By the hypothesis, $\langle a \rangle$ is s-c-permutably embedded in G. Hence $T/N = \langle a \rangle N/N$ is s-c-permutably embedded in G/N by Lemma 2.1. Analogously we can see that every primary cyclic subgroup of BN/N is s-c-permutably embedded in G/N. Hence G/N satisfies the hypothesis. The minimal choice of G implies that G/N is supersoluble.

(3) $\Phi(G) = 1$ and G has a unique minimal normal subgroup N such that $N = C_G(N) = O_p(G) = F(G)$ is a Sylow subgroup of G and $G = [N]M$, where M is a maximal subgroup of G with $M_p = 1$.

Since the class of all supersoluble groups is a saturated formation, by (2), we see that G has a unique minimal normal subgroup N and $\Phi(G) = 1$. 1. Hence there exists a maximal subgroup M of G such that N

$\not\subseteq M$ and so $G = NM$. Since G is soluble, N is an elementary abelian p-subgroup of G for some prime p. Obviously, $N \cap M \lhd G$ and so $N \cap M = 1$. It follows that $G = [N]M$. Let $C = C_G(N)$. Obviously, $C \cap M \trianglelefteq G$. If $C \cap M \neq 1$, then $N \subseteq C \cap M \subseteq M$, a contradiction. Hence $C \cap M = 1$. By Dedekind identity, $C = C \cap NM = N(C \cap M) = N$. This implies that $N = O_p(G) = F(G) = C_G(N)$. By [3, Lemma 1.7.11], $O_p(M) = 1$. On the other hand, since G' is nilpotent, $G' \leq F(G) = N$. Consequentle $N = G'$. Therefore $M \cong G/N = G/G'$ is abelian and thus $M_p = 1$ since $O_p(M) = 1$. It follows that N is a Sylow p-subgroup of G.

(4) Final contradiction.

Since $G = AB$ and the unique minimal normal subgroup N is a Sylow p-subgroup of G, $A \cap N \neq 1$ or $B \cap N \neq 1$. Without loss of generality, we assume that $A \cap N = A_p \neq 1$. Let C_p be some cyclic subgroup of A_p with order p. By the hypothesis and Lemma 2.3, C_p is s-c-permutable in G. Let q be an arbitrary prime of $|G|$ with $p \neq q$. Then there exists a Sylow q-subgroup Q of G such that $C_pQ = QC_p.$. Since N is an elementary abelian p-subgroup and $N \trianglelefteq G, C_p \trianglelefteq N$ and C_p is a subnormal Hall subgroup of C_pQ. It follows that $Q \leq N_G(C_p)$. The arbitrary choice of q implies that $C_p \trianglelefteq G$. Hence $|N| = |C_p| = p$ and consequently G is supersoluble. The final contradiction completes the proof.

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